TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 339, Number 2, October 1993

ZEROS OF THE SUCCESSIVE DERIVATIVES OF HADAMARD GAP SERIES

ROBERT M. GETHNER

ABSTRACT. A complex number z is in the *final set* of an analytic function f, as defined by Pólya, if every neighborhood of z contains zeros of infinitely many $f^{(n)}$. If f is a Hadamard gap series, then the part of the final set in the open disk of convergence is the origin along with a union of concentric circles.

1. INTRODUCTION

A complex number z is in the *final set* $\Lambda(f)$ of an analytic function f if every neighborhood of z contains zeros of infinitely many $f^{(n)}$. Final sets of various functions have been determined by Pólya [4, 5] (who introduced the notion) and others (see [2] for references). A power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^{N_k}, \tag{1.1}$$

with $c_k \neq 0$ for all k, has *Hadamard gaps* if there exists $L > 1$ such that

$$N_{k+1}/N_k > L \quad \text{for all } k \geq 0. \tag{1.2}$$

Theorem 1. *Let f be a function whose Maclaurin series has Hadamard gaps and (finite or infinite) radius of convergence R. Then $\Lambda(f) \cap \{|z| < R\} = \{0\} \cup \{z : |z| \in E\}$, where E is closed in the topology of $(0, R)$.*

Theorem 1 is best possible in the following sense.

Theorem 2. *Let R be in $(0, \infty]$, and let E be closed in the topology of $(0, R)$. Then there exists a Hadamard gap series f with radius of convergence R such that $\Lambda(f) \cap \{|z| < R\} = \{0\} \cup \{|z| : z \in E\}$.*

I am grateful to L. R. Sons and W. H. J. Fuchs for advice and encouragement.

2. PROOF OF THEOREM 1

The proof of Theorem 1 depends on two lemmas, which I will prove in §§3 and 4, respectively, concerning functions h of the form

$$h(z) = \sum_{k=0}^{\infty} a_k z^{n_k}. \tag{2.1}$$

Received by the editors November 16, 1990 and, in revised form, July 16, 1991.
1991 *Mathematics Subject Classification.* Primary 30D35, 30B10.
Key words and phrases. Final set, successive derivatives, Hadamard gaps, gap series.

Fix such an h and denote by R the radius of convergence of the series. Set

$$\mu(r) = \max\{|a_k|r^{n_k} : k \geq 0\}, \qquad \nu(r) = \max\{k : |a_k|r^{n_k} = \mu(r)\}. \tag{2.2}$$

(This notation is not standard; see [6, p. 3].) Finally, call a number r in $(0, R)$ *h-dominant* if

$$\sum_{k=0}^{\nu(r)-1} |a_k|r^{n_k} + \sum_{k=\nu(r)+1}^{\infty} n_k^{\nu(r)}|a_k|r^{n_k} < \mu(r), \tag{2.3}$$

where the first sum is taken to be zero if $\nu(r) = 0$.

The first lemma is an adaptation of [3, Theorem 6, p. 605]. Denote by $Z(s, t, \theta_1, \theta_2)$ the number of zeros (counting multiplicity) of h in the set $\{re^{i\theta} : s \leq r \leq t \text{ and } \theta_1 \leq \theta \leq \theta_2\}$.

Lemma 1. *Let h have the form* (2.1) (*not necessarily with Hadamard gaps*), *and let R, $\mu(r)$, and $\nu(r)$ be as above. If s and t are h-dominant, if $s < t$, and if $0 < \theta_2 - \theta_1 < 2\pi$, then*

$$\left| Z(s, t, \theta_1, \theta_2) - (n_{\nu(t)} - n_{\nu(s)})\frac{\theta_2 - \theta_1}{2\pi} \right| < \nu(t) + 2.$$

Lemma 2. *Let h and R be as in Lemma* 1, *and suppose that there exists $L > 1$ such that $n_{k+1}/n_k > L$ for all $k \geq 0$. Suppose also that*

$$n_0 \geq \max\{9, \exp[\sqrt{(\log 6)(\log L)}]\}. \tag{2.4}$$

Define

$$\tau = 54e^{-2}/(\log L)(1 - 1/L)(1 - L^{-1/3}). \tag{2.5}$$

Then there is at least one h-dominant point in each interval $(C, D) \subset (0, R)$ such that

$$\log(D/C) > \tau/n_0^{1/3}. \tag{2.6}$$

Proof of Theorem 1. $0 \in \Lambda(f)$ by (1.2).

Define h_j, $a_k = a_k(j)$, and $n_k = n_k(j)$ by

$$h_j(z) = z^j f^{(j)}(z) = \sum_{k=0}^{\infty} a_k z^{n_k}. \tag{2.7}$$

Then by (1.2),

$$\text{(a)} \quad n_{k+1}/n_k > L > 1, \qquad \text{(b)} \quad n_0 \geq j. \tag{2.8}$$

Define a set $E \subset (0, R)$ as follows: $r^* \in E$ if there exist an infinite set T of positive integers and a sequence $\{r_j\}_{j\in T}$ such that

$$\text{(a)} \quad \lim_{j\to\infty, j\in T} r_j = r^*, \qquad \text{(b)} \quad \text{no } r_j \text{ is } h_j\text{-dominant}. \tag{2.9}$$

I will show that if $r^* \in E$ then $\{|z| = r^*\} \subset \Lambda(f)$, whereas if $r^* \notin E$ then $\{|z| = r^*\} \cap \Lambda(f)$ is empty.

Case I. $r^* \in E$. Choose $\{r_j\}$ as above and define τ by (2.5). By (2.8b) and (2.9a), $r_j \exp\{2\tau/n_0^{1/3}\} < R$ for all large j in T. Pick such a $j > \max\{9, \exp[\sqrt{(\log 6)(\log L)}]\}$. By Lemma 2, there are h_j-dominant points $s = s(j)$ in $(r_j \exp\{-2\tau/n_0^{1/3}\}, r_j)$ and $t = t(j)$ in $(r_j, r_j \exp\{2\tau/n_0^{1/3}\})$.

Then

$$\nu(s, h_j) < \nu(t, h_j). \tag{2.10}$$

For suppose that $\nu(s) = \nu(t) \equiv p$, and set

$$\psi(r) = \frac{1}{|a_p|r^{n_p}} \left(\sum_{k=0}^{p-1} |a_k|^{r^{n_k}} + \sum_{k=p+1}^{\infty} n_k^p |a_k| r^{n_k} \right).$$

Then $\psi(s) < 1$ and $\psi(t) < 1$ by (2.3). Hence $\psi(r_j) < 1$ since ψ is convex [7, p. 172]. Thus r_j is h_j-dominant, contrary to the definition of r_j. This proves (2.10).

Put

$$U_j(\theta_1, \theta_2) = \{re^{i\theta} : r_j \exp\{-2\tau/n_0^{1/3}\} \le r \le r_j \exp\{2\tau/n_0^{1/3} \text{ and } \theta_1 \le \theta \le \theta_2\}.$$

I will show that, if j is sufficiently large, then

(2.11)

h_j has at least one zero in $U_j(\theta_1, \theta_2)$ whenever $\theta_2 - \theta_1 > 6\pi L^{-j}/(1 - L^{-1})$.

For $xL^{-x} \downarrow$ for large x, so that, when $j \le k$, (2.8) and (2.10) give $k/n_k < kL^{-k}/j \le L^{-j}$ and $(n_{\nu(t)} - n_{\nu(s)})/n_{\nu(t)} > 1 - L^{-[\nu(t)-\nu(s)]} \ge 1 - L^{-1}$; thus, by Lemma 1, the number of zeros in $U_j(\theta_1, \theta_2)$ is at least

$$\begin{aligned} Z(s, t, \theta_1, \theta_2) &\ge (n_{\nu(t)} - n_{\nu(s)}) \frac{\theta_2 - \theta_1}{2\pi} - \nu(t) - 2 \\ &> (1 - L^{-1}) n_{\nu(t)} \frac{\theta_2 - \theta_1}{2\pi} - 3\nu(t) \\ &> 3n_{\nu(t)} L^{-j} - 3\nu(t) > 3\nu(t) - 3\nu(t) = 0, \end{aligned}$$

which establishes (2.11).

Now by (2.9a) and (2.8b), $r_j \exp\{-2\tau/n_0^{1/3}\} \to r^*$ and $r_j \exp\{2\tau/n_0^{1/3}\} \to r^*$ as $j \to \infty$ in T. Thus (2.11) implies that every point of $\{|z| = r^*\}$ is a limit point of zeros of $\{h_j\}_{j \in T}$, so that, by (2.7), $\{|z| = r^*\} \subset \Lambda(f)$.

Case II. $r^* \notin E$. For all large j and small ε, r is h_j-dominant for r in $I \equiv (r^* - \varepsilon, r^* + \varepsilon)$. So by (2.3),

$$|h_j(z)| \ge \mu(r, h_j) - \sum_{k=0}^{\nu(r, h_j)-1} |a_k| r^{n_k} - \sum_{k=\nu(r, h_j)+1}^{\infty} |a_k| r^{n_k} > 0$$

whenever $|z| = r \in I$. This completes the proof of Theorem 1.

3. Proof of Lemma 1

We need two more lemmas. The first is a variation on [6, Problem 66, p. 45]; the second is an adaptation of [3, Lemma 7]. Let D denote differentiation.

Lemma 3. *Let $J \equiv (a, b) \subset \mathbb{R}^+$, let $g: J \to \mathbb{C}$ be differentiable, and let $\alpha \in \mathbb{Z}^+$. If $\operatorname{Im}\{g\}$ changes sign at least twice in J, then $\operatorname{Im}\{(rD - \alpha)g\}$ changes sign there at least once.*

Proof. For real r,

$$\operatorname{Im}\{(rD - \alpha)g(r)\} = \operatorname{Im}\left\{ r^{\alpha+1} \frac{d}{dr}[r^{-\alpha} g(r)] \right\} = r^{\alpha+1} \frac{d}{dr}[r^{-\alpha} \operatorname{Im}\{g(r)\}],$$

and the lemma follows from Rolle's Theorem.

For a function H analytic on a contour C, denote by $\Delta(H, C)$ the variation over C of any continuous branch of $\arg H$.

Lemma 4. *Let h have the form (2.1) with radius of convergence R, let $[s, t] \subset (0, R)$, and suppose that t is h-dominant. Set $I = I(\theta) = \{re^{i\theta} : s \le r \le t\}$. If $h \ne 0$ on I, then $|\Delta(h, I)| \le \pi[\nu(t) + 1]$.*

Proof. We may assume that $\theta = 0$. Set $q = \nu(t)$. Choose ϕ so that $e^{i\phi} a_q$ is positive imaginary, and put

$$H(r) = e^{i\phi}(rD - n_0)(rD - n_1)\cdots(rD - n_{q-1})h(r) \equiv \sum_{k=n_q}^{\infty} b_k r^{n_k}, \tag{3.1}$$

where

$$b_k = (n_k - n_0)(n_k - n_1)\cdots(n_k - n_{q-1})e^{i\phi}a_k. \tag{3.2}$$

I claim that $\operatorname{Im}\{H\}$ does not change sign in (s, t). If the claim is correct, then q applications of Lemma 3 show that $\operatorname{Im}\{h\}$ changes sign at most q times in (s, t), so that the curve $h(I)$ crosses the real axis at most q times. Therefore $|\Delta(h, I)| \le \pi(q + 1)$, and the lemma follows.

To prove the claim, pick r in (s, t) and set

$$\psi(r) = \frac{1}{|b_q| r^{n_q}} \left(\sum_{k=n_{q+1}}^{\infty} |b_k| r^{n_k} \right).$$

We have $|b_k| \le n_k^q |a_k|$ and $|b_q| \ge |a_q|$ by (3.2). Thus, since t is h-dominant and $q = \nu(t)$, (2.3) gives

$$\psi(t) \le \frac{1}{|a_q| t^{n_q}} \left(\sum_{k=n_{q+1}}^{\infty} n_k^q |a_k| t^{n_k} \right) = \frac{1}{\mu(t)} \left(\sum_{k=n_{\nu(t)+1}}^{\infty} n_k^{\nu(t)} |a_k| t^{n_k} \right) < 1.$$

Now ψ increases, so, by (3.1),

$$|H(r) - b_q r^{n_q}| \le \sum_{k=q+1}^{\infty} |b_k| r^{n_k} = \psi(r) |b_q| r^{n_q} < |b_q| r^{n_q}.$$

But our choice of ϕ makes $b_q r^{n_q}$ positive imaginary, so that $H(r)$ is in the upper half-plane. This establishes the claim and Lemma 4.

Proof of Lemma 1. Let $\Gamma = I_1 \cup C_t \cup I_2 \cup C_s$, where $I_1 = \{re^{i\theta_1} : s \le r \le t\}$, $I_2 = \{re^{i\theta_2} : s \le r \le t\}$, $C_s = \{se^{i\theta} : \theta_1 \le \theta \le \theta_2\}$, and $C_t = \{te^{i\theta} : \theta_1 \le \theta \le \theta_2\}$. Also put $P(z) = a_{\nu(s)} z^{n_{\nu(s)}}$ and $Q(z) = a_{\nu(t)} z^{n_{\nu(t)}}$.

First assume that $h \ne 0$ on Γ. Then

$$\begin{aligned} &\Delta(h, \Gamma) - \Delta(P, C_s) - \Delta(Q, C_t) \\ &\quad = \Delta(h/P, C_s) + \Delta(h/Q, C_t) + \Delta(h, I_1) + \Delta(h, I_2). \end{aligned} \tag{3.3}$$

Also, (2.3) gives $|h(z)/P(z) - 1| < 1$, and hence $\operatorname{Re}\{h(z)/P(z)\} > 0$, for $z \in C_s$. So $|\Delta(h/P, C_s)| \le \pi$. Similarly, $|\Delta(h/Q, C_t)| \le \pi$. Thus, by (3.3) and Lemma 4,

$$|\Delta(h, \Gamma) - (n_{\nu(t)} - n_{\nu(s)})(\theta_2 - \theta_1)| \le 2\pi + 2\pi[\nu(t) + 1], \tag{3.4}$$

and Lemma 1 follows from the argument principle. If h has zeros on Γ, apply (3.4) to a nearby contour Γ' on which $h \neq 0$ and let $\Gamma' \to \Gamma$.

4. Proof of Lemma 2

Lemma 5. *Let h, R, and L be as in Lemma 2, and let* (2.4) *hold. Pick $m \geq 0$ and $[A, B] \subset (0, R)$, and suppose that*

$$|a_k|s^{n_k} \leq |a_m|s^{n_m} \quad \text{for all } k \geq 0 \text{ and } s \text{ in } [A, B] \tag{4.1}$$

and

$$\log\frac{B}{A} > \frac{6}{(\log L)(1-1/L)}\frac{(\log n_m)^2}{n_m}. \tag{4.2}$$

Then there exists r in (A, B) such that (2.3) *holds with $\nu(r) = m$.*

Proof of Lemma 2. For each $m \geq 0$, set $I_m = \{r \geq 0\colon |a_m|r^{n_m} = \mu(r, h)\}$. Denote by (A, B) the interior of $I_m \cap (C, D)$. If (A, B) has no h-dominant points, then (4.2) must fail. Therefore, since $\bigcup I_m = \mathbb{R}^+$,

$$\begin{aligned}\log\frac{D}{C} &= \int_C^D \frac{dx}{x} = \sum_{m=0}^{\infty}\int_{I_m \cap (C, D)} \frac{dx}{x} \\ &\leq \frac{6}{(\log L)(1-1/L)}\sum_{m=0}^{\infty}\frac{(\log n_m)^2}{n_m}.\end{aligned} \tag{4.3}$$

Now $(\log x)/x^{1/3} \leq 3/e$ for $x > 0$; also, $n_m^{1/3} > (L^m n_0)^{1/3}$ by (2.8a). Thus $(\log n_m)^2/n_m \leq 9e^{-2}n_m^{2/3}/n_m < 9e^{-2}(L^m n_0)^{-1/3}$. So by (4.3) and (2.5),

$$\log\frac{D}{C} \leq \frac{1}{n_0^{1/3}}\frac{6}{(\log L)(1-1/L)}\frac{9}{e^2}\sum_{m=0}^{\infty} L^{-m/3} = \frac{\tau}{n_0^{1/3}}.$$

But this contradicts (2.6), and the proof is complete.

Proof of Lemma 5. Set

$$\sigma = \exp\{(\log n_m)^2/(n_m \log L)\}. \tag{4.4}$$

Then

$$\sigma > 1, \tag{4.5}$$

$$n_k^m \leq \sigma^{n_k} \quad \text{for all } k \geq m, \tag{4.6}$$

$$\text{(a)}\quad 2\sigma^{n_m}\{\sigma(A/B)^{1/2}\}^{(1-1/L)n_m} < \tfrac{1}{3}, \qquad \text{(b)}\quad \sigma(A/B)^{1/2} < 1. \tag{4.7}$$

Proof of (4.6). By (2.8a) and (2.4), $k < (\log n_k)/(\log L)$. Also, $(\log x)^2/x$ decreases for $x > e^2$. Hence, by (4.4),

$$\begin{aligned} m\log n_k \leq k\log n_k &\leq \frac{(\log n_k)^2}{\log L} = \frac{(\log n_k)^2}{n_k}\frac{n_k}{\log L} \\ &\leq \frac{(\log n_m)^2}{n_m}\frac{n_k}{\log L} = n_k\log\sigma. \end{aligned}$$

Proof of (4.7). By (4.2) and (4.4), $(A/B)^{(1-1/L)n_m/2} < \sigma^{-3n_m}$. By (2.4), $\sigma^{n_m} \geq \sigma^{n_0} \geq 6$. Therefore

$$\{\sigma(A/B)^{1/2}\}^{(1-1/L)n_m} < \sigma^{(1-1/L)n_m - 3n_m} < \sigma^{-n_m}\sigma^{-n_m} \leq \sigma^{-n_m}/6.$$

This yields (4.7a), and (b) follows from (a) and (4.5).

We are now ready to prove (2.3) with $\nu(r) = m$ and

$$r = (AB)^{1/2}. \tag{4.8}$$

When $k \geq m+1$, (2.8a) implies that

$$n_k - n_m = \sum_{\gamma=m+1}^{k} (n_\gamma - n_{\gamma-1}) \geq \sum_{\gamma=m+1}^{k} (L-1)n_{\gamma-1} \geq (L-1)n_m(k-m). \tag{4.9}$$

By (4.6), (4.1) with $s = B$, and (4.8), and by (4.9), (which we may apply because of (4.7b)),

$$\begin{aligned} n_k^m|a_k|r^{n_k} &\leq |a_m|B^{n_m}\{\sigma(A/B)^{1/2}\}^{n_k} \\ &\leq |a_m|r^{n_m}\sigma^{n_m}\{\sigma(A/B)^{1/2}\}^{(L-1)}n_m(k-m). \end{aligned} \tag{4.10}$$

Next, if $0 \leq k < m-1$, then (2.8a) gives

$$n_k \leq n_{m-1} = n_m - (n_m - n_{m-1}) < n_m - (1-1/L)n_m.$$

Thus, by (4.1) with $s = A$ and (4.8),

$$|a_k|r^{n_k} \leq |a_m|A^{n_m}\{(B/A)\}^{1/2}\}^{n_k} \leq |a_m|r^{n_m}\{(A/B)^{1/2}\}^{(1-1/L)n_m}. \tag{4.11}$$

We have $m \leq n_m \leq \sigma^{n_m}$ from (2.4) and (4.6). Also, $L-1 > 1-1/L$ by (2.8a). Thus (4.11), (4.10), (4.5), and (4.7a) give

$$\begin{aligned} &\sum_{k=0}^{m-1}|a_k|r^{n_k} + \sum_{k=m+1}^{\infty} n_k^m|a_k|r^{n_k} \\ &\quad < |a_m|r^{n_m}\left\{m\left[\sigma\left(\frac{A}{B}\right)^{1/2}\right]^{(1-1/L)n_m} + \sigma^{n_m}\frac{[\sigma(A/B)^{1/2}]^{(1-1/L)n_m}}{1-[\sigma(A/B)^{1/2}]^{(1-1/L)n_m}}\right\} \\ &\quad \leq |a_m|r^{n_m}\left\{\frac{[2\sigma^{n_m}][\sigma(A/B)^{1/2}]^{(1-1/L)n_m}}{1-[\sigma(A/B)^{1/2}]^{(1-1/L)n_m}}\right\} < |a_m|r^{n_m}\frac{1/3}{1-1/6} < |a_m|r^{n_m}. \end{aligned}$$

This yields (2.3) and completes the proof of Lemma 5.

5. Proof of Theorem 2

The following construction is similar to that in [1].

Proof of Theorem 2. Pick positive sequences $\{r_P\}_{P=0}^{\infty}$, $\{R_P\}_{P=0}^{\infty}$, and $\{\varepsilon_P\}_{P=0}^{\infty}$ so that

(5.1)
- (a) the set of limit points of $\{r_P\}$ in $(0, R)$ is E,
- (b) $r_P e^{\varepsilon_P} < R_P < R$,
- (c) $\varepsilon_P \to 0$,
- (d) $R_P \to R$.

Choose a function $\psi\colon \mathbb{Z}^+ \to \mathbb{R}^+$ such that

$$[\psi(x)]^{1/x} \downarrow 1/R \quad \text{as } x \to \infty. \tag{5.2}$$

Define f by (1.1), where $\{c_k\}$ and $\{N_k\}$ are defined inductively as follows. Set

$$\text{(a)}\ \ N_0 = 3, \qquad \text{(b)}\ \ c_{-1} = 1. \tag{5.3}$$

Having chosen $c_{-1}, \ldots, c_{2P-1} > 0$ and $N_0, \ldots, N_{2P}$, pick $c_{2P}, c_{2P+1}, N_{2P+1}$, and N_{2P+2} as follows. Using (5.2) and (5.1b), pick N_{2P+1} large enough so that

$$\text{(5.4)} \qquad \text{(a)} \quad N_{2P+1}/N_{2P} > 2, \qquad \text{(b)} \quad N_{2P+1}^{-N_{2P}} > (e^{-\varepsilon_P/2})^{N_{2P+1}-N_{2P}},$$

$$\text{(5.5)} \qquad N_{2P+1} \log\{[\psi(N_{2P+1})]^{1/N_{2P+1}} R_P\} < -2\log 3,$$

and

$$\text{(5.6)} \qquad \{[\psi(N_{2P+1})]^{1/N_{2P+1}} r_P\}^{N_{2P+1}/N_{2P}} r_P^{-1} < c_{2P-1}^{1/N_{2P-1}}.$$

Set

$$\text{(5.7)} \qquad \text{(a)} \quad c_{2P+1} = \psi(N_{2P+1}), \qquad \text{(b)} \quad c_{2P} = c_{2P+1} r_P^{N_{2P+1}-N_{2P}}.$$

Finally, pick N_{2P+2} large enough so that

$$\text{(5.8)} \qquad \text{(a)} \quad N_{2P+2}/N_{2P+1} > 2, \qquad \text{(b)} \quad 2N_{2P+1}^2 \frac{\log N_{2P+2}}{N_{2P+2}} < \log 3.$$

The function f just constructed satisfies

$$\text{(5.9)} \qquad c_{2P+1}^{1/N_{2P+1}} \geq c_k^{1/N_k} \quad \text{for all } P \geq 0 \text{ and } k \geq 2P+1.$$

For odd k, (5.9) follows from (5.7a) and (5.2). If $k \equiv 2Q$ is even, (5.9) follows from (5.7) and (5.6) (both with P replaced by Q):

$$c_{2Q}^{1/N_{2Q}} = (c_{2Q+1}^{1/N_{2Q+1}} r_Q)^{N_{2Q+1}/N_{2Q}} r_Q^{-1} < c_{2Q-1}^{1/N_{2Q-1}} \leq c_{2P+1}^{1/N_{2P+1}}.$$

f has radius of convergence R by (5.7a), (5.2), and (5.9). By (1.1) and (5.8a), $0 \in \Lambda(f)$. We need the following lemma, proved at the end of the paper, to show that $\Lambda(f) \cap \{0 < |z| < R\} = \{z: |z| \in E\}$.

Lemma 6. *Let f be as above, and define ϕ by*

$$\text{(5.10)} \qquad f^{(j)}(z) = \sum_{N_k \geq j} c_k N_k (N_k - 1) \cdots (N_k - j + 1) z^{N_k - j} \equiv \sum_{N_k \geq j} \phi_{jk}(z).$$

If $P \geq 0$, and if

$$\text{(5.11)} \qquad \text{(a)} \quad j \leq N_{2P+1} \quad \textit{and} \quad \text{(b)} \quad |z| \leq R_P,$$

then

$$\text{(5.12)} \qquad \sum_{k=2P+2}^{\infty} |\phi_{jk}(z)| \leq |\phi_{j,2P+1}(z)|/2.$$

Proof that $\Lambda(f) \cap \{0 < |z| < R\} \subset \{z: |z| \in E\}$. By (5.1), it is enough to show that $f^{(j)}(z) \neq 0$ whenever either

$$\text{(5.13)} \qquad j \in (N_{2P}, N_{2P+1}] \quad \text{and} \quad 0 < |z| \leq R_P$$

or

$$\text{(5.14)} \quad j \in (N_{2P-1}, N_{2P}] \quad \text{and} \quad z \in \{0 < |z| \leq r_P e^{-\varepsilon_P}\} \cup \{r_P e^{\varepsilon_P} \leq |z| \leq R_P\}.$$

But if (5.13) holds, then (5.10) and (5.12) give

$$|f^{(j)}(z)| \geq |\phi_{j,2P+1}(z)| - \sum_{k=2P+2}^{\infty} |\phi_{jk}(z)| \geq |\phi_{j,2P+1}(z)|/2 > 0.$$

If (5.14) holds, define

$$G(j, P) = \frac{N_{2P}(N_{2P}-1)\cdots(N_{2P}-j+1)}{N_{2P+1}(N_{2P+1}-1)\cdots(N_{2P+1}-j+1)}. \tag{5.15}$$

Then $G(j, P) > 1/N_{2P+1}^{j} > N_{2P+1}^{-N_{2P}}$, so that, by (5.4b),

$$(e^{-\varepsilon_P/2})^{N_{2P+1}-N_{2P}} < G(j, P) < 1. \tag{5.16}$$

Also, by (5.10), (5.15), and (5.7b),

$$\left|\frac{\phi_{j,2P}(z)}{\phi_{j,2P+1}(z)}\right| = \frac{c_{2P}}{c_{2P+1}} G(j, P)|z|^{N_{2P}-N_{2P+1}} = G(j, P)\left(\frac{r_P}{|z|}\right)^{N_{2P+1}-N_{2P}}. \tag{5.17}$$

If $r_P e^{\varepsilon_P} \le |z| \le R_P$, then $|\phi_{j,2P}(z)/\phi_{j,2P+1}(z)| < (e^{-\varepsilon_P})^{N_{2P+1}-N_{2P}} \le 3^{-6}$ by (5.17), (5.16), (5.4b), and (5.3). Hence, by (5.10) and (5.12),

$$\begin{aligned} |f^{(j)}(z)| &\ge \phi_{j,2P+1}(z)| - \sum_{k=2P+2}^{\infty} |\phi_{jk}(z)| - |\phi_{j,2P}(z)| \\ &\ge |\phi_{j,2P+1}(z)| - |\phi_{j,2P+1}(z)|/2 - |\phi_{j,2P+1}(z)|/3^6 > 0. \end{aligned}$$

Similarly, if $0 < |z| \le r_P e^{-\varepsilon_P}$, then

$$|\phi_{j,2P}(z)/\phi_{j,2P+1}(z)| > (e^{\varepsilon_P/2})^{N_{2P+1}-N_{2P}} \ge 3^3$$

and

$$\begin{aligned} |f^{(j)}(z)| &\ge |\phi_{j,2P}(z)| - \sum_{k=2P+2}^{\infty} |\phi_{jk}(z)| - |\phi_{j,2P+1}(z)| \\ &\ge |\phi_{j,2P}(z)| - \left(\frac{3}{2}\frac{1}{3^3}\right)|\phi_{j,2P+1}(z)| > 0. \end{aligned}$$

Proof that $\{z: |z| \in E\} \subset \Lambda(f)$. Fix P and set

$$r = [G(2P, P)]^{1/(N_{2P+1}-N_{2P})} r_P. \tag{5.18}$$

Then, by (5.16),

$$r \in (r_P e^{-\varepsilon_P}, r_P). \tag{5.19}$$

Set $h_j(z) = z^j f^{(j)}(z)$. For $|z| = r$, we have

$$|\phi_{2P,2P}(z)| = |\phi_{2P,2P+1}(z)| > \sum_{k=2P+2}^{\infty} |\phi_{2P,k}(z)|$$

by (5.18), (5.17), and (5.12). Thus, by (5.10) and (2.2),

$$\mu(r, h_{2P}) = |z^{2P}\phi_{2P,2P}(z)| = |z^{2P}\phi_{2P,2P+1}(z)|.$$

Therefore r violates the definition (2.3) of h_{2P}-dominance. It now follows from (5.19) and (5.1ca) that E is in the set of limit points of the points which are not h_{2P}-dominant. Thus $\{z: |z| \in E\} \subset \Lambda(f)$ by the paragraph containing (2.9). This completes the proof of Theorem 2.

Proof of Lemma 6. Pick $k \ge 2P+2$. By (5.9) and (5.7a),

$$\begin{aligned} \log\frac{c_k}{c_{2P+1}} &= \frac{\log c_k}{N_k} N_k - \frac{\log c_{2P+1}}{N_{2P+1}} N_{2P+1} \\ &\le (N_k - N_{2P+1})\frac{\log \psi(N_{2P+1})}{N_{2P+1}}. \end{aligned} \tag{5.20}$$

Also, $(\log x)/x$ decreases for $x > N_{2P+1}$ by (5.3a). Thus, by (5.8),

$$
\begin{aligned}
N_{2P+1}\frac{\log N_k}{N_k - N_{2P+1}} &= \frac{N_{2P+1}}{1 - N_{2P+1}/N_k}\frac{\log N_k}{N_k} \\
&< \frac{N_{2P+1}}{1/2}\frac{\log N_{2P+2}}{N_{2P+2}} \leq \frac{\log 3}{N_{2P+1}}.
\end{aligned}
\tag{5.21}
$$

By (5.10), (5.11), (5.20), (5.21), and (5.5),

$$
\begin{aligned}
\log\left|\frac{\phi_{jk}(z)}{\phi_{j,2P+1}(z)}\right| &\leq \log\frac{c_k}{c_{2P+1}} + N_{2P+1}\log N_k + (N_k - N_{2P+1})\log R_P \\
&\leq (N_k - N_{2P+1})\left[\log(\{\psi(N_{2P+1})\}^{1/N_{2P+1}} R_P) + \frac{\log 3}{N_{2P+1}}\right] \\
&< (N_k - N_{2P+1})\left(\frac{-\log 3}{N_{2P+1}}\right).
\end{aligned}
\tag{5.22}
$$

But $(N_k - N_{2P+1})/N_{2P+1} \geq k - 2P - 1$ by (5.4a), (5.8a), and (4.9) (with $L = 2$, N_k in place of n_k, and $m = 2P + 1$). Thus (5.22) gives

$$
\sum_{k=2P+2}^{\infty}\left|\frac{\phi_{jk}(z)}{\phi_{j,2P+1}(z)}\right| \leq \sum_{k=2P+2}^{\infty} e^{-(\log 3)(k-2P-1)} = \frac{1/3}{1-1/3} = \frac{1}{2}.
$$

References

1. A. Edrei and G. R. Maclane, *On the zeroes of the derivatives of an entire function*, Proc. Amer. Math. Soc. **8** (1957), 702–706.
2. R. M. Gethner, *Zeros of the successive derivatives of Hadamard gap series in the unit disk*, Michigan Math. J. **36** (1989), 403–414.
3. W. K. Hayman, *Angular value distribution of power series with gaps*, Proc. London Math. Soc. (3) **24** (1972), 590–624.
4. G. Pólya, *Über die Nullstellen sukzessiver Derivierten*, Collected Papers (R. P. Boas, ed.), MIT Press, Cambridge, Mass., 1974.
5. ——, *On the zeros of the derivatives of a function and its analytic character*, Collected Papers (R. P. Boas, ed.), MIT Press, Cambridge, Mass., 1974.
6. G. Pólya and G. Szegö, *Problems and theorems in analysis*, vol. 2, Springer-Verlag, New York, 1972.
7. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, Oxford, 1939.

Department of Mathematics, Franklin and Marshall College, Lancaster, Pennsylvania 17604-3003

E-mail address: r_gethner@acad.fandm.edu